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Pulsed Instabilities in Solid-Propellant Rockets

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Occurrences of pulsed instabilities in rocket combustion chambers have long posed irritating practical and puzzling theoretical questions. The term "triggering" was applied to the phenomenon in the 1960s to describe observations and computations of the unstable steep-fronted waves generated by sufficiently large pulses injected in linearly stable gas and liquid rockets. All such instabilities are classified as subcritical bifurcations in the theory of dynamical systems. Understanding the physical reasons for the existence of subcritical bifurcations in combustion chambers, and therefore the conditions under which they will occur, has been the subject of many investigations in the last three decades. It has long been recognized that the most likely causes of pulsed instabilities must be associated either with nonlinear gasdynamics or with nonlinear combustion processes. With numerical analysis of longitudinal oscillations, Baum and Levine have convincingly shown that nonlinear combustion is required. By suitable adjustment of parameters in a simple representation of the response of a burning solid, they have shown quite good agreement between computations and many experimental results. The results reported in this article are consistent with those of Baum and Levine, establishing the existence of pulsed instabilities when both nonlinear gasdynamics and nonlinear combustion processes are accounted for.

Nomenclature

A_b	=	admittance function for the burning surface
A_n, B_n	=	amplitudes, Eq. (12)
A_{nij}, B_{nij}	=	nonlinear gasdynamic coefficients, Eq. (30)
a	=	speed of sound
$C_{ni}^{(1)}$, $C_{ni}^{(2)}$	=	nonlinear gasdynamic coefficients, Eq. (31)
D	=	diameter of chamber
$D_{ni}^{(1)}$, $D_{ni}^{(2)}$	=	nonlinear gasdynamic coefficients, Eq. (31)
E_n^2	=	inner product defined by Eq. (9b)
F_n	=	forcing function of the nth acoustic mode
F	=	Eq. (2)
f	=	nonlinear functional arising from the boundary conditions
h	=	nonlinear functional arising from conservation equations
$I_{nij}^{(A)}$, $I_{ni}^{(B)}$, $I_{nij}^{(C)}$	=	nonlinear combustion coefficients, Eqs. (34a–34c)
k_n	=	wave number of the nth acoustic mode
L	=	length of chamber
M_b	=	u_b/a, Mach number at the burning surface
M_r	=	characteristic Mach number
$\dot{m}$	=	mass flux at the surface
$\hat{n}$	=	unit normal vector
P	=	Eq. (2)
p	=	pressure
R_b	=	linear pressure coupled response function
R_{vc}	=	velocity coupled response function
$\bar{R}_{vc}$	=	$R_{vc}a$, nondimensional velocity coupled response function
r_n	=	amplitude, Eq. (12)
$\boldsymbol{r}$	=	spatial coordinates
S	=	surface area
T_s	=	temperature of burning surface
t	=	time
u	=	velocity
V	=	volume
x	=	longitudinal coordinate
$\bar{x}$	=	$\pi x/L$, nondimensional coordinate
α_n	=	linear growth rate of the nth acoustic mode
γ	=	ratio of specific heats
δ_{mn}	=	Kronecker delta
η_n	=	amplitude of the nth acoustic mode
θ_n	=	frequency shift of the nth acoustic mode
ρ	=	density
τ_1	=	period of the fundamental mode
ϕ_n	=	phase, Eq. (12)
ψ_n	=	mode shape of the nth acoustic mode
ω_n	=	frequency of the nth acoustic mode

Subscripts

b	=	at the burning surface
pc	=	pressure coupled
vc	=	velocity coupled

Superscripts

BL	=	Baum and Levine's model
G	=	Greene's model
GD	=	gasdynamics
(i)	=	imaginary part of a complex quantity
NL	=	nonlinear
(r)	=	real part of a complex quantity
$\bar{\ }$	=	mean quantity
$'$	=	fluctuating quantity
$\dot{\ }$	=	time derivative

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I. Introduction

IT is useful to divide the problems of unsteady motions in combustion chambers into three classes: 1) linear stability, 2) linear instability and the conditions for existence and stability of limit cycles, and 3) nonlinear or pulsed instability. By far, most work has been concerned with linear stability, but beginning in the mid-1960s, increasing attention has been directed to the nonlinear behavior associated with linear instabilities. Of the three classes of problems, pulsed instabilities are the least well understood, although their practical and theoretical importance has been recognized from the beginning of work on nonlinear behavior.

The three classes of problems are listed in order of increasing difficulty of both analysis and understanding. Both experimental and theoretical methods for treating linear instabilities in solid rockets are highly developed and are used routinely,[1–3] though comparisons of predictions and observed behavior are not entirely satisfactory. However, experience has shown that the errors and uncertainties are typically due to imperfect information about the response of combustion to unsteady motions and other contributing processes.

Limit cycles for thermally driven waves were first treated by Chu[4] and Chu and Ying,[5] using the method of characteristics. Combustion processes and average flow were not included, although the heat source was allowed to vary with pressure. Using similar techniques, Sirignano[6] and Sirignano and Crocco[7] first accounted for some of the possible influences of combustion and flow. In those and subsequent works[8–10] by the Princeton group, the problems of the existence of stable limit cycles and conditions under which pulsing to stable or unstable limit cycles can occur were first exposed. However, the nature of the methods used did not easily produce the conditions for existence and stability of limit cycles. Additionally, restrictive approximations to the combustion processes and flowfield were necessary since all of the analyses were concerned essentially with solving partial differential equations (PDEs) in space and time.

Zinn and his students,[8–10] and independently, Culick,[11,12] began use of a form of Galerkin's method to investigate combustion instabilities. The great advantages of that approach are as follows:

1) Spatial averaging provides a framework that easily accommodates realistic models for all processes.

2) Analysis involves solving ordinary nonlinear differential equations, for which recent advances in dynamical systems theory have become available.

Considerable progress has been achieved with this approach, producing, e.g., analytical results for the limit cycles when a small number of modes is present.[13–17] In addition, limited results for pulsed instabilities have been produced by Kim[18] and Greene[19] using time-averaging and truncation to two modes. The results reported here are the first, however, for pulsed instabilities found within the theory based on Galerkin's method without those additional approximations. Most previous works have assumed that only gasdynamics provided nonlinear influences. What we have established (by example, not by formal proof), is that gasdynamics to the order treated so far does not contain pulsed instabilities. In this work we assume that the combustion process is nonlinear, a contribution that will show the possibility of "triggering," i.e., pulsing a linearly stable system into stable limit cycles.

The unstable growth of oscillations induced by finite pulses is potentially serious in practical rocket motors. During the Apollo program, NASA developed a method of rating the stability of liquid rocket engines by "bombing," i.e., exploding small charges during a firing. A motor is then considered acceptably stable if the decay rate exceeds a chosen value. Liquid rockets may experience pulses, e.g., during ignition, changes of thrust level, or if liquid reactants accumulate and suddenly ignite. A common cause for pulses in solid rockets is expulsion of pieces of an igniter or of insulation. Motivated by the operational implications of such events, the Air Force Rocket Propulsion Laboratory conducted a lengthy experimental and theoretical program beginning in the mid-1970s. Baum et al.[20] published the last paper covering that project through 1988.

To understand the observations, Baum and Levine solved the PDEs numerically for conditions approximating those in the tests. To obtain agreement between predicted behavior and observations, parameters representing the unsteady combustion response were suitably changed. As described in Sec. III, a simple model of nonlinear combustion response was used, based on the idea of a kinematical nonlinearity associated with velocity coupling.

Although quite satisfactory results were obtained, the authors were well aware of the deficiencies of using a purely numerical analysis to interpret measurements. It is, e.g., difficult to make definite conclusions concerning the relative influences of nonlinear combustion and other processes, although the results showed convincingly that nonlinear combustion is an essential feature of those pulsed instabilities treated. Moreover, because the run time of the numerical solutions was limited by practical considerations arising with the computers then available, it is not clear that any of the numerical results definitely represent stable limit cycles for long times after the pulses. As with any numerical work, each calculation is a special case: extracting qualitative rules-of-thumb potentially helpful in design is a tedious matter. Finally, extension to two- and three-dimensional problems has yet to be accomplished, largely because of cost; in principle, much more complicated problems could now be treated.

The chief purpose of the work reported here is to investigate pulsed nonlinear instabilities using the approximate analysis based on Galerkin's method. We are not attempting at this point to obtain agreement between predictions and experimental data. Rather, our intention is to clarify various aspects of the method and particularly to emphasize its advantages for investigating general characteristics of nonlinear instabilities without the large costs of time and computation accompanying full numerical analysis of the problem. (However, as emphasized in Ref. 1 and elsewhere, accurate numerical solutions are essential as the only means for assessing the accuracy of approximate results.)

II. Formulation of the Approximate Analysis

The analysis used here has been described elsewhere, beginning in 1976[12] and recently in two summary works.[1,21] The formulation is quite general, capable of accommodating any geometry and all physical processes. Previous work[15–17,21] has shown that in several important respects, the results for third-order acoustics are qualitatively like those for second-order acoustics. In particular, calculations for a wide range of cases have shown that nonlinear gasdynamics does not contain pulsed instabilities. Consequently, we treat acoustics only to second-order, leading to a nonlinear wave equation for the pressure fluctuation

$$\nabla^2 p' - \frac{1}{a^2}\frac{\partial^2 p'}{\partial t^2} = h \tag{1}$$

where a is taken to be constant and

$$\begin{aligned} h = &-\bar{\rho}\nabla\cdot(\bar{\boldsymbol{u}}\cdot\nabla\boldsymbol{u}' + \boldsymbol{u}'\cdot\nabla\bar{\boldsymbol{u}}) + \frac{1}{a^2}\bar{\boldsymbol{u}}\cdot\nabla\frac{\partial p'}{\partial t} + \frac{\gamma}{a^2}\frac{\partial p'}{\partial t}\nabla\cdot\bar{\boldsymbol{u}} \\ &- \nabla\left(\rho\boldsymbol{u}'\cdot\nabla\boldsymbol{u}' + \rho'\frac{\partial\boldsymbol{u}'}{\partial t}\right) + \frac{1}{a^2}\frac{\partial}{\partial t}(\boldsymbol{u}'\cdot\nabla p') \\ &+ \frac{\gamma}{a^2}\frac{\partial}{\partial t}(p'\nabla\cdot\boldsymbol{u}') + \nabla\cdot\boldsymbol{F}' - \frac{1}{a^2}\frac{\partial P'}{\partial t} \end{aligned} \tag{2}$$

The boundary condition on p' is

$$\hat{\boldsymbol{n}}\cdot\nabla p' = -f \tag{3}$$

$$\begin{aligned} f = &\bar{\rho}\frac{\partial\boldsymbol{u}'}{\partial t}\cdot\hat{\boldsymbol{n}} + \bar{\rho}(\bar{\boldsymbol{u}}\cdot\nabla\boldsymbol{u}' + \boldsymbol{u}'\cdot\nabla\bar{\boldsymbol{u}})\cdot\hat{\boldsymbol{n}} \\ &+ \bar{\rho}(\boldsymbol{u}'\cdot\nabla\boldsymbol{u}')\cdot\hat{\boldsymbol{n}} + \rho'\frac{\partial\boldsymbol{u}'}{\partial t}\cdot\hat{\boldsymbol{n}} - \boldsymbol{F}'\cdot\hat{\boldsymbol{n}} \end{aligned} \tag{4}$$

$\boldsymbol{F}'$ and P' in Eqs. (2) and (4) contain all sources other than those explicitly shown, such as combustion processes within the volume and interactions between the gas and condensed material. We assume that the contributions in h and f are small and treat them as perturbations to the classical linear acoustics problem defined by setting $h = f = 0$. For this analysis we use the eigenfunctions, the classical normal modes ψ_n, defined by the homogeneous problem

$$\nabla^2\psi_n + k_n^2\psi_n = 0 \tag{5a}$$

$$\hat{\boldsymbol{n}}\cdot\nabla\psi_n = 0 \tag{5b}$$

Spatial averaging is applied by multiplying Eq. (1) by ψ_n, Eq. (5a) by p', subtracting the results and integrating over the chamber. After the use of Green's theorem to introduce the boundary condition, we have the equations

$$-\frac{1}{a^2}\int \psi_n \frac{\partial^2 p'}{\partial t^2}\,\mathrm{d}V - k_n^2\int \psi_n p'\,\mathrm{d}V = \int \psi_n h\,\mathrm{d}V + \int \psi_n f\,\mathrm{d}S \tag{6}$$

We may use the normal modes as an orthogonal basis for an approximation to the unsteady pressure field

$$p'(\boldsymbol{r}, t) = \bar{p}\sum_{m=1}^{\infty} \eta_m(t)\psi_m(\boldsymbol{r}) \tag{7}$$

where the time-dependent amplitude $\eta_m(t)$ will eventually be the variables to be determined. All terms in h and f are either second-order in fluctuations, or linear multiplied by another small quantity (e.g., the velocity or Mach number of the mean flow). Hence, as a first approximation in h and f, it is adequate to use Eq. (7) and the corresponding formula for the acoustic velocity

$$\boldsymbol{u}'(\boldsymbol{r}, t) = \sum_{m=1}^{\infty} \frac{\dot{\eta}_m(t)}{\gamma k_m^2}\nabla\psi_m(\boldsymbol{r}) \tag{8}$$

Using these approximations, the actual boundary conditions are of course not satisfied. However, h and f are small and the influence of the violation of the boundary condition is confined to a region near the surface that is a small part of the volume. Thus, the expansions (7) and (8), although zeroth-order, can be used legitimately to determine the perturbations of the flowfield to first-order. [However, the use of (8) implies that the fluctuating velocity in the chamber is associated entirely with acoustical waves, defined by the connection between velocity and pressure fluctuations. This is a significant approximation relaxed in a work in progress.] Also, the conditions for pulsed instabilities to exist will be valid to first-order when using these approximations. It is possible, but rarely necessary, to obtain results valid to higher order using an iterative procedure explained in Refs. 11 and 21.

We assume that the unperturbed normal modes are orthogonal so that

$$\int \psi_m\psi_n\,\mathrm{d}V = E_n^2\delta_{mn} \tag{9a}$$

$$E_n^2 = \int \psi_n^2\,\mathrm{d}V \tag{9b}$$

Substitution of Eq. (7) in Eq. (6) and use of Eqs. (9a) and (9b) leads to the set of coupled nonlinear equations:

$$\frac{\mathrm{d}^2\eta_n}{\mathrm{d}t^2} + \omega_n^2\eta_n = F_n \tag{10}$$

where $\omega_n = ak_n$ and

$$F_n = -\frac{a^2}{\bar{p}E_n^2}\left(\int \psi_n h\,\mathrm{d}V + \oint \psi_n f\,\mathrm{d}S\right) \tag{11}$$

Generally, F_n depends both linearly and nonlinearly on the pressure and velocity fluctuations. For acoustic motions, it is appropriate to use Eqs. (7) and (8) to evaluate F_n, thereby introducing η_n in the right-hand side (RHS) of Eq. (10). More general unsteady motions require further consideration, as will be discussed in Sec. III.

In previous work based on this analysis, considerable use has been made of the first-order equations produced by time-averaging Eq. (10). The idea is that in many applications, the oscillations have slowly varying amplitudes and phases: only small fractional changes in one period of the lowest mode. It is then reasonable to write (without approximation at this point)

$$\eta_n(t) = r_n(t)\sin[\omega_n t + \phi_n(t)] = A_n(t)\sin\,\omega_n t + B_n(t)\cos\,\omega_n t \tag{12}$$

Substitution in Eq. (10) and averaging over one period τ_1 of the fundamental mode gives[21]

$$\frac{\mathrm{d}A_n}{\mathrm{d}t} = \frac{1}{\omega_n}\int_t^{t+\tau_1} F_n\cos\,\omega_n t'\,\mathrm{d}t' \tag{13a}$$

$$\frac{\mathrm{d}B_n}{\mathrm{d}t} = -\frac{1}{\omega_n}\int_t^{t+\tau_1} F_n\sin\,\omega_n t'\,\mathrm{d}t' \tag{13b}$$

A_n and B_n in F_n are treated as constants during integration since they vary little in the time τ_1. One purpose of the present work is to determine the effects of time-averaging by comparison of solutions to Eqs. (13a) and (13b) with those of the original oscillator equations (10).

III. Modeling Nonlinear Unsteady Combustion of a Solid Propellant

Combustion of a solid propellant is nonlinear chiefly for two reasons:

1) Chemical processes depend nonlinearly on both temperature and pressure.

2) The conversion of condensed material to gaseous products is a nonlinear function of the properties of the local flowfield.

Nonlinear behavior necessarily arises in any representation of the chemical kinetics, the strongest influence being due to the Arrhenius factor appearing in the usual formulas for the reaction rate. One possible nonlinear effect is that the burning rate of a solid may be dependent on changes in the magnitude, but not the direction of the flow past the surface. This is commonly referred to as "velocity coupling." For the purposes here, the precise formula for nonlinear unsteady combustion is unimportant. We intend primarily to show that nonlinear combustion is sufficient, within the analysis described above, to cause pulsed instabilities. We investigate the matter with two models, the first due to Baum and Levine,[20,22] which has been successfully used in comparisons of numerical results and observations of pulsed instabilities. The second model, proposed by Greene,[19] is also based on the idea of velocity coupling, but using a slightly different form.

To be used in the present analysis, any model of unsteady combustion must be put in such a form as to fit into the appropriate terms in the force given by Eq. (11). Use of h and f in Eq. (11) leads to the formula F_n to second-order:

$$-\frac{\bar{p}E_n^2}{a^2}F_n = \left\{\bar{\rho}\int(\bar{\boldsymbol{u}}\cdot\nabla\boldsymbol{u}' + \boldsymbol{u}'\cdot\nabla\bar{\boldsymbol{u}})\cdot\nabla\psi_n\,\mathrm{d}V + \frac{1}{a^2}\frac{\partial}{\partial t}\int(\gamma p'\nabla\cdot\bar{\boldsymbol{u}} + \bar{\boldsymbol{u}}\cdot\nabla p')\psi_n\,\mathrm{d}V\right\}$$

linear gasdynamics

$$+\left\{\bar{\rho}\int\left[\boldsymbol{u}'\cdot\nabla\boldsymbol{u}' + \frac{\rho'}{\bar{\rho}}\frac{\partial\boldsymbol{u}'}{\partial t}\right]\cdot\nabla\psi_n\,\mathrm{d}V\right.$$

$$+ \frac{1}{a^2}\frac{\partial}{\partial t}\int (\gamma p' \nabla\cdot \boldsymbol{u}' + \boldsymbol{u}'\cdot\nabla p')\psi_n \, \mathrm{d}V\Bigg\}$$

nonlinear gasdynamics

$$+ \left\{\int \bar{\rho}\frac{\partial \boldsymbol{u}'}{\partial t}\cdot\hat{\boldsymbol{n}}\psi_n \, \mathrm{d}S\right\} - \int\left[\frac{1}{a^2}\frac{\partial P'}{\partial t}\psi_n + \boldsymbol{F}'\cdot\nabla\psi_n\right]\mathrm{d}V \tag{14}$$

linear and nonlinear surface processes — other contributions

The linear and nonlinear gasdynamics terms will be computed using the zeroth-order approximations (7) and (8) for the pressure and velocity. To the same order, the density fluctuation may be set equal to its isentropic value, $\rho'/\bar{\rho} = p'/\gamma\bar{p}$, legitimate because we assume that there is no residual combustion within the chamber. The terms labeled "other contributions" will be ignored.

Nonlinear combustion can be included in the term representing surface processes by use of the mass flux at the surface. [We replace the natural boundary condition involving $\bar{\rho}(\partial\boldsymbol{u}'/\partial t)\cdot\hat{\boldsymbol{n}}$ by (15) mainly to accommodate the models previously introduced by others and used here.] By definition, $\dot{\boldsymbol{m}} = \rho\boldsymbol{u}$, so

$$\begin{aligned}\dot{\boldsymbol{m}}' &= \dot{\boldsymbol{m}} - \bar{\dot{\boldsymbol{m}}} \\ &= \rho\boldsymbol{u}' + \rho'(\bar{\boldsymbol{u}} + \boldsymbol{u}')\end{aligned}$$

Because $\bar{\boldsymbol{u}}$ is independent of time, a form that can be directly substituted in Eq. (14) is obtained by taking the time derivative and rearranging terms:

$$\bar{\rho}\frac{\partial \boldsymbol{u}'}{\partial t}\cdot\hat{\boldsymbol{n}} = \frac{\partial \dot{\boldsymbol{m}}'}{\partial t}\cdot\hat{\boldsymbol{n}} - \rho'\frac{\partial \boldsymbol{u}'}{\partial t}\cdot\hat{\boldsymbol{n}} - \frac{\partial \rho'}{\partial t}(\bar{\boldsymbol{u}} + \boldsymbol{u}')\cdot\hat{\boldsymbol{n}} \tag{15}$$

Using Eq. (15) in the surface process term in Eq. (14) yields

$$\begin{aligned}-\frac{\bar{p}E_n^2}{a^2}(F_n)_{\mathrm{comb}} &= \int\frac{\partial \dot{\boldsymbol{m}}'}{\partial t}\cdot\hat{\boldsymbol{n}}\psi_n\,\mathrm{d}S \\ &- \int\rho'\frac{\partial \boldsymbol{u}'}{\partial t}\cdot\hat{\boldsymbol{n}}\psi_n\,\mathrm{d}S - \int\frac{\partial\rho'}{\partial t}(\bar{\boldsymbol{u}}+\boldsymbol{u}')\cdot\hat{\boldsymbol{n}}\psi_n\,\mathrm{d}S\end{aligned} \tag{16}$$

The last two integrals are evaluated by writing $\rho' = \bar{\rho}(p'/\gamma\bar{p})$ and using the linear approximation for $\boldsymbol{u}'$ expressed in terms of the admittance function[1] A_b:

$$-\boldsymbol{u}'\cdot\hat{\boldsymbol{n}} = aA_b(p'/\gamma\bar{p}) \tag{17}$$

Substitution in Eq. (16) gives

$$\begin{aligned}-\frac{\bar{p}E_n^2}{a^2}(F_n)_{\mathrm{comb}} &= -\int\frac{\partial \dot{m}'}{\partial t}\psi_n\,\mathrm{d}S \\ &+ \bar{\rho}\int\left(\bar{u}_b + 2aA_b\frac{p'}{\gamma\bar{p}}\right)\frac{\partial}{\partial t}\left(\frac{p'}{\gamma\bar{p}}\right)\psi_n\,\mathrm{d}S\end{aligned} \tag{18}$$

where we have written $\bar{u}_b = -\bar{\boldsymbol{u}}\cdot\hat{\boldsymbol{n}}$ for the average velocity of the flow inward at the burning surface and $\dot{\boldsymbol{m}}'\cdot\hat{\boldsymbol{n}} = -\dot{m}'$ for the fluctuation of inward mass flux.

A. Baum and Levine's Model

This is an ad hoc model in which the mass burning rate is directly modified by some function of the velocity. For that reason it was originally called the "burn rate augmentation model." The total mass burning rate is written as a combination of linear pressure coupling and nonlinear velocity coupling as

$$\dot{m} = \dot{m}_{\mathrm{pc}}[1 + R_{\mathrm{vc}}F(\boldsymbol{u})] \tag{19}$$

where $\dot{m}_{\mathrm{pc}}$ is the mass flux due to pressure only and R_{vc} is a constant related to the sensitivity of burning to velocity parallel to the surface. Since the evolution rate of solid to gas should depend on the magnitude, but not the direction, of the scouring flow, the simplest assumption is to take $F(\boldsymbol{u})$ equal to $|\boldsymbol{u}'|$, ignoring the possible influence of mean flow speed on mass flux:

$$\dot{m} = \dot{m}_{\mathrm{pc}}[1 + R_{\mathrm{vc}}|\boldsymbol{u}'|] \tag{20}$$

The fluctuation of $\dot{m}$ is calculated with Eq. (20)

$$\begin{aligned}\dot{m}' &= \dot{m} - \bar{\dot{m}} \\ &= \dot{m}_{\mathrm{pc}}'[1 + R_{\mathrm{vc}}|\boldsymbol{u}'|] + \bar{\dot{m}}_{\mathrm{pc}}R_{\mathrm{vc}}|\boldsymbol{u}'|\end{aligned}$$

By definition of the response function R_b for linear pressure coupling[1]

$$\dot{m}_{\mathrm{pc}}'/\bar{\dot{m}} = R_b(p'/\gamma\bar{p}) \tag{21}$$

and we have

$$\dot{m}'/\bar{\dot{m}} = R_b(p'/\gamma\bar{p}) + R_bR_{\mathrm{vc}}(p'/\gamma\bar{p})|\boldsymbol{u}'| + (\bar{\dot{m}}_{\mathrm{pc}}/\bar{\dot{m}})R_{\mathrm{vc}}|\boldsymbol{u}'| \tag{22}$$

Substitution of Eq. (22) in Eq. (18) gives

$$\begin{aligned}\frac{\bar{p}E_n^2}{a^2}(F_n)^{\mathrm{BL}}_{\mathrm{comb}} &= \frac{\bar{\dot{m}}R_bR_{\mathrm{vc}}}{\gamma}\int\frac{\partial}{\partial t}\left[\frac{p'}{\bar{p}}|\boldsymbol{u}'|\right]\psi_n\,\mathrm{d}S \\ &+ \bar{\dot{m}}_{\mathrm{pc}}R_{\mathrm{vc}}\int\frac{\partial|\boldsymbol{u}'|}{\partial t}\psi_n\,\mathrm{d}S - \frac{2\bar{\rho}aA_b}{\gamma^2}\int\left(\frac{p'}{\bar{p}}\right)\frac{\partial}{\partial t}\left(\frac{p'}{\bar{p}}\right)\psi_n\,\mathrm{d}S \\ &+ \left\{\frac{\bar{\dot{m}}}{\gamma}\int(R_b - 1)\frac{\partial}{\partial t}\left(\frac{p'}{\bar{p}}\right)\psi_n\,\mathrm{d}S\right\}\end{aligned}$$

For simplicity, assume that there is no steady erosive burning, so that $\bar{\dot{m}}_{\mathrm{pc}} = \bar{\dot{m}} = \bar{\rho}\bar{u}_b$ and drop the linear terms in braces by combining them with other linear processes. Then the formula for F_n due to combustion is

$$\begin{aligned}(F_n)^{\mathrm{BL}}_{\mathrm{comb}} &= \frac{\bar{\dot{m}}a^2}{\bar{p}E_n^2}\left\{\frac{R_bR_{\mathrm{vc}}}{\gamma}\int\frac{\partial}{\partial t}\left(\frac{p'}{\bar{p}}|\boldsymbol{u}'|\right)\psi_n\,\mathrm{d}S\right. \\ &\left. + R_{\mathrm{vc}}\int\frac{\partial|\boldsymbol{u}'|}{\partial t}\psi_n\,\mathrm{d}S - \frac{2A_b}{\gamma^2\bar{M}_b}\int\left(\frac{p'}{\bar{p}}\right)\frac{\partial}{\partial t}\left(\frac{p'}{\bar{p}}\right)\psi_n\,\mathrm{d}S\right\}\end{aligned} \tag{23}$$

where $\bar{M}_b = \bar{\dot{m}}/\bar{\rho}a = \bar{u}_b/a$.

With a simple assumption we can reduce the number of parameters by one. From the definition of A_b and R_b [Culick and Yang,[1] Eq. (93)]

$$A_b + \bar{M}_b = \bar{M}_b\left(R_b + \gamma\frac{\Delta T_s/\bar{T}_s}{\eta_n/\psi_n}\right)$$

For the purposes here it is sensible to ignore nonisentropic fluctuations of temperature in the flame, set $\Delta T_s = 0$, and

$$A_b/\bar{M}_b = R_b - 1 \tag{24}$$

Hence, Eq. (23) becomes

$$(F_n)^{BL}_{comb} = \frac{\bar{u}_b}{E_n^2}\left\{C_1 \int \frac{\partial}{\partial t}\left(\frac{p'}{\bar{p}}|\boldsymbol{u}'|\right)\psi_n \, dS + C_2 \int \frac{\partial |\boldsymbol{u}'|}{\partial t}\psi_n \, dS - C_3 \int \frac{p'}{\bar{p}}\frac{\partial}{\partial t}\left(\frac{p'}{\bar{p}}\right)\psi_n \, dS\right\} \tag{25}$$

with

$$C_1 = R_b R_{vc}, \qquad C_2 = R_{vc}/\gamma, \qquad C_3 = (2/\gamma)(R_b - 1) \tag{26}$$

It is a well-known result that linear calculations of the response function for sinusoidal motions produce the result that R_b is a complex quantity, $R_b = R_b^{(r)} + iR_b^{(i)}$. Here, we will set $R_b^{(i)} = 0$ so that C_1 and C_2 are real constants. Such an arbitrary choice is within the intent of this paper to investigate the qualitative behavior, in particular to determine whether or not triggering can be found.

B. Greene's Model

Greene[19] carried out calculations with a response dependent on the absolute value of the velocity

$$\dot{m}' = \bar{\dot{m}} C_4 |\boldsymbol{u}'| \tag{27}$$

Substitution in Eq. (18) and omission of the linear terms as in Sec. III.A gives the result corresponding to Eq. (23)

$$(F_n)^{G}_{comb} = \frac{\bar{u}_b}{E_n^2}\left\{C_4 \int \frac{\partial |\boldsymbol{u}'|}{\partial t}\psi_n \, dS - C_3 \int \left(\frac{p'}{\bar{p}}\right)\frac{\partial}{\partial t}\left(\frac{p'}{\bar{p}}\right)\psi_n \, dS\right\} \tag{28}$$

Thus, Greene's model leads to two of the three terms appearing in Baum and Levine's model represented by Eq. (25).

IV. Equations for Computing the Time-Dependent Pressure Field

Because the procedure for evaluating the integrals arising from the linear and nonlinear gasdynamics has been previously documented in several places,[1,12,18] only a brief summary is required here. First, all linear processes, those arising from the gasdynamics and any others generated from P' and $\boldsymbol{F}'$, will produce the terms $2\alpha_n\dot{\eta}_n + 2\omega_n\theta_n\eta_n$ in the nth equation of the set (10). In general, linear coupling will also be present, although in principle, it can be eliminated by defining a new set of orthogonal modes by applying familiar methods. In any case, the effects of linear coupling are of order $\bar{M}_r^2$. Since the system of equations (10) with F_n given by Eq. (14) involves the assumption that terms of order $\bar{M}_r^2$ and higher are not included, linear coupling terms must be dropped. We will assume here that any linear coupling arising from P' and $\boldsymbol{F}'$ is also negligible. Hence, the system (10) has the form

$$\ddot{\eta}_n + \omega_n^2\eta_n = 2\alpha_n\dot{\eta}_n + 2\omega_n\theta_n\eta_n + (F_n)_{nonlinear} \tag{29}$$

If we follow the practice of previous works and drop terms of order $\bar{M}_r(M_r')^2$, where M_r' is a characteristic Mach number for the unsteady flow, then the gasdynamics to second-order, those terms shown in the formula (14), lead to[12]

$$(F_n)^{GD}_{nonlinear} = -\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}[A_{nij}\dot{\eta}_i\dot{\eta}_j + B_{nij}\eta_i\eta_j] \tag{30}$$

Setting the last term of Eq. (29) equal to the sum of Eq. (30) and either Eq. (25) or (28) gives the set of oscillator equations applicable to a chamber of any shape. For solid-propellant rockets, problems of pulsed instabilities arise only in motors having relatively high L/D, and involve mainly axial fluctuations. Therefore, for the numerical results discussed in Sec. V, only longitudinal modes will be treated. The double sum in Eq. (30) then becomes a single sum and the equations are[15]

$$\ddot{\eta}_n + \omega_n^2\eta_n = 2\alpha_n\dot{\eta}_n + 2\omega_n\theta_n\eta_n - \sum_{i=1}^{n-1}[C_{ni}^{(1)}\dot{\eta}_i\dot{\eta}_{n-i} + D_{ni}^{(1)}\eta_i\eta_{n-i}] - 2\sum_{i=1}^{\infty}[C_{ni}^{(2)}\dot{\eta}_i\dot{\eta}_{n+i} + D_{ni}^{(2)}\eta_i\eta_{n+i}] + (F_n)^{NL}_{comb} \tag{31}$$

where $(F_n)^{NL}_{comb}$ is given either by Eq. (25) or (28) and

$$\begin{aligned} C_{ni}^{(1)} &= \frac{-1}{2\gamma i(n-i)}[n^2 + i(n-i)(\gamma - 1)] \\ D_{ni}^{(1)} &= \frac{(\gamma-1)\omega_1^2}{4\gamma}[n^2 - 2i(n-i)] \\ C_{ni}^{(2)} &= \frac{1}{2\gamma i(n+i)}[n^2 - i(n+i)(\gamma - 1)] \\ D_{ni}^{(2)} &= \frac{(\gamma-1)\omega_1^2}{4\gamma}[n^2 + 2i(n+i)] \end{aligned} \tag{32}$$

V. Discussion of Results

Dynamical systems theory has proven to be a very useful tool in the study of systems of ordinary differential equations (ODEs). Although this theory has been used extensively in other fields, it has only recently been applied to the study of nonlinear combustion instabilities (first by Jahnke and Culick[23]). When applied to the system of equations derived in the approximate analysis, dynamical systems theory provides a systematic approach for locating the possibility of pulsed instabilities and is, therefore, well suited to the present analysis. In particular, periodic solutions may be traced as a function of one free parameter of the system using a continuation method. For our purposes, the free parameter is chosen to be the linear growth rate of the fundamental mode, and the maximum amplitudes of the acoustic modes in limit cycle are plotted as a function of this parameter. As shown in Table 1, the linear growth rates of all other modes are negative so that when $\alpha_1 < 0$, the system is linearly stable. For descriptions of the continuation method, see Jahnke and Culick[23] and Doedel et al.[24,25]

The continuation method described earlier will be used to study longitudinal modes in a cylindrical chamber. Specifically, the chamber has a length of 0.5969 m and a radius of 0.0253 m giving $L/D = 11.8$. The fundamental frequency is 5654.86 Hz. These values along with the linear parameters

Table 1 Linear growth rates and frequency shifts

	1	2	3	4	5	6
α_i, s^{-1}	Free	−324.8	−583.6	−889.4	−1262.7	−1500.0
θ_i, rad/s	12.9	46.8	−29.3	−131.0	−280.0	−300.0

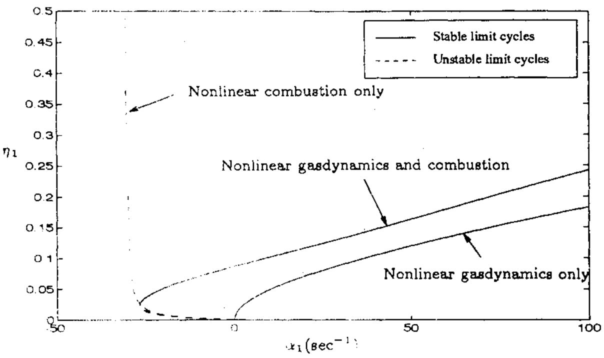

Fig. 1 Maximum amplitude of first acoustic mode in limit cycle showing contributions of nonlinear gasdynamics and combustion.

from Table 1 are used to investigate nonlinearities leading to pulsed instabilities.

Although there is no rigorous proof, extensive previous results (including those obtained by Jahnke and Culick) have convincingly established that within the analysis used here nonlinear gasdynamics does not contain pulsed instabilities. Therefore, this work concentrates on the effects of nonlinear combustion. Both nonlinear combustion models adopted here can generate subcritical bifurcations, and hence the possibility for pulsed instabilities, but only if nonlinear gasdynamics is also accounted for. In Fig. 1, Greene's model is used with Eq. (31) to illustrate the contributions of nonlinear gasdynamics and combustion.

We will now consider Baum and Levine's model. Recall that Greene's model (28) appears as two of the three terms in Eq. (25). Substitution of Eqs. (7) and (8) in Eq. (25) yields

$$(F_n)^{\mathrm{BL}}_{\mathrm{comb}} = \frac{2(L/D)}{\gamma\pi^2}\frac{\bar{u}_b}{a} \times \Bigg\{ \underbrace{-2R_b\bar{R}_{\mathrm{vc}} \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left[\frac{\dot{\eta}_i\dot{\eta}_j}{i} - i\omega_1^2\eta_i\eta_j\right] I^{(A)}_{nij}}_{A} + \underbrace{\frac{4\omega_1^2\bar{R}_{\mathrm{vc}}}{\gamma}\sum_{i=1}^{\infty} i\eta_i I^{(B)}_{ni}}_{B} - \underbrace{4\omega_1(R_b-1)\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\eta_i\dot{\eta}_j I^{(C)}_{nij}}_{C}\Bigg\} \tag{33}$$

where $\bar{R}_{\mathrm{vc}} = R_{\mathrm{vc}}a$ and

$$I^{(A)}_{nij} = \int_0^{\pi} \cos n\tilde{x}[\sin(i-j)\tilde{x} + \sin(i+j)\tilde{x}]\operatorname{sign}(u')\,\mathrm{d}\tilde{x} \tag{34a}$$

$$I^{(B)}_{ni} = \int_0^{\pi} \cos n\tilde{x}\sin i\tilde{x}\operatorname{sign}(u')\,\mathrm{d}\tilde{x} \tag{34b}$$

$$I^{(C)}_{nij} = \int_0^{\pi} \cos n\tilde{x}[\cos(i-j)\tilde{x} + \cos(i+j)\tilde{x}]\,\mathrm{d}\tilde{x} \tag{34c}$$

with $\tilde{x} = \pi x/L$. Thus, Baum and Levine's model consists of three different terms. One purpose of the following calculations is to determine what effect each term has on the system. In particular, we would like to answer the question: which terms are sufficient to produce triggering for reasonable values of $\bar{R}_{\mathrm{vc}}$ and/or R_b? Another purpose of the present analysis is to determine the effects of truncation to a small number of modes and time-averaging, two approximations often used in previous works.

A. Effects of Individual Terms

The three terms of the Baum and Levine model, labeled A, B, and C in Eq. (33), represent three different types of nonlinear combustion response. By using each of these terms individually with the original oscillator equations (31), their effect and relative importance will be determined.

The first term in the Baum and Levine model, term A, represents coupling to both pressure and velocity oscillations, and thus, depends on both R_b and $\bar{R}_{\mathrm{vc}}$. Figure 2 shows that for truncation to two and four modes, this term produces the possibility of triggering, shown by a region of stable limit cycles for $\alpha_1 < 0$. A value of 2.18 was used for the linear pressure coupled response function R_b. This value was used in the study by Baum et al.[20] and seems to be a reasonable choice. Because it was not clear how the velocity coupled response function R_{vc}, was nondimensionalized in that work, however, we could not determine what value of $\bar{R}_{\mathrm{vc}}$ to use for comparison with their results. Instead, $\bar{R}_{\mathrm{vc}} = 16.15$ was chosen to give a sizable region of triggering, although this value seems high.

If the value of $\bar{R}_{\mathrm{vc}}$ is increased even higher, it was found that eventually, the velocity coupling becomes so strong that triggering is no longer possible. In order to explain this observation, a two-parameter continuation using $\bar{R}_{\mathrm{vc}}$ as a second free parameter was employed to plot the loci of turning points or folds. The result is shown in Fig. 3. As $\bar{R}_{\mathrm{vc}}$ is increased from zero, a turning point defining the lower limit of possible triggering is created at $\alpha_1 = 0\ \mathrm{s}^{-1}$ and shifts to the left. The upper turning point also shifts to the left as $\bar{R}_{\mathrm{vc}}$ is increased. At a value of $\bar{R}_{\mathrm{vc}} = 22.8$, the loci of turning points meet. Above this value, the possibility of triggering does not exist. The four-mode case produced similar results, although the loci met at a higher value of $\bar{R}_{\mathrm{vc}}$. It will be shown later that for more reasonable values of $\bar{R}_{\mathrm{vc}}$, the term A, the first term in Eqs. (25) and (33), will have a rather small effect on the

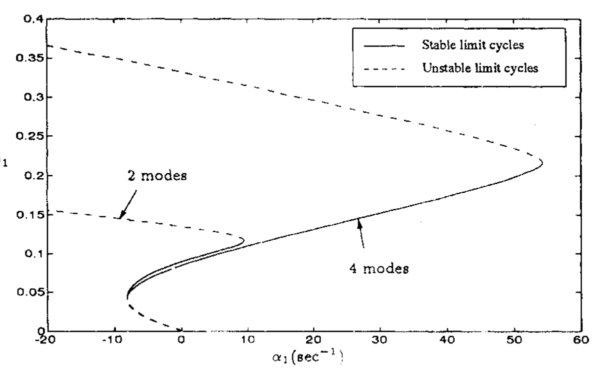

Fig. 2 Maximum amplitude of first acoustic mode using term A with the original oscillator equations; two and four modes; $\bar{R}_{\mathrm{vc}} = 16.15$, $R_b = 2.18$.

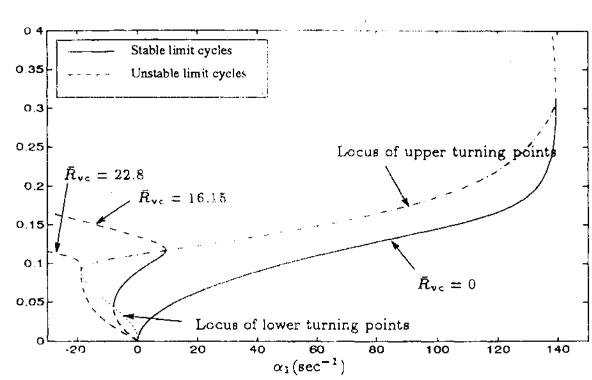

Fig. 3 Loci of turning point bifurcations for the first acoustic mode, original oscillator equations using term A; two modes; $R_b = 2.18$.

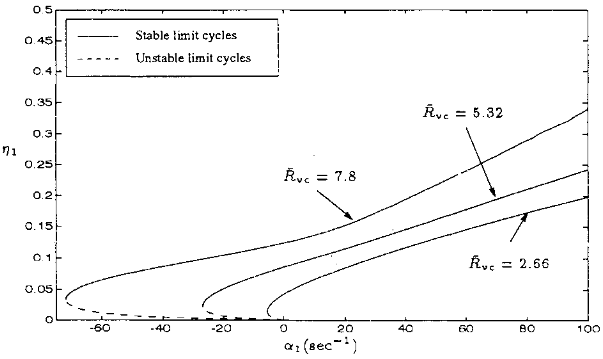

Fig. 4 Variation of triggering region with respect to $\bar{R}_{vc}$ using term B with the original oscillator equations; four modes.

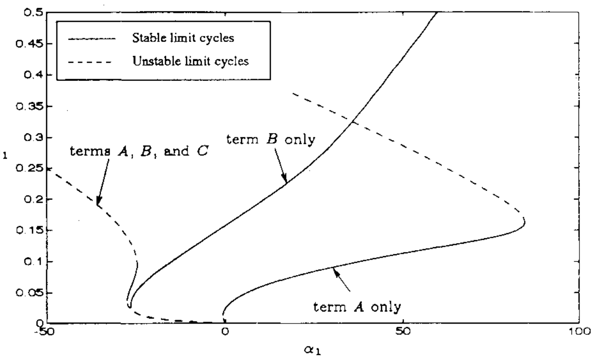

Fig. 5 Comparison of contribution of terms A and B from Baum and Levine's nonlinear combustion model; two modes; $\bar{R}_{vc}$ = 5.32, R_b = 2.18.

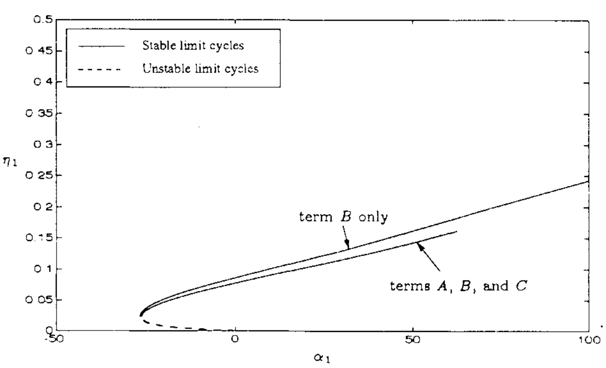

Fig. 6 Comparison of contribution from all terms vs term B from Baum and Levine's nonlinear combustion model; four modes; $\bar{R}_{vc}$ = 5.32, R_b = 2.18.

system. We do not have a physical explanation for the behavior shown in Fig. 3.

The second term in Eq. (33), labeled B, represents coupling to velocity oscillations only. This term gives the possibility of triggering to stable limit cycles for more reasonable values of $\bar{R}_{vc}$, as shown in Fig. 4 for $\bar{R}_{vc}$ = 2.66, 5.32, and 7.8. By comparing Figs. 3 and 4, it can be noted that for R_b = 2.18, as well as other practical values of R_b, this term produces a much larger region of triggering for smaller values of $\bar{R}_{vc}$. Indeed, this is the expected behavior since term B is first-order in perturbations, whereas term A is second-order.

The final term, C in Eq. (33), is a nonlinear pressure coupling term. For very high values of R_b, this term produced a subcritical bifurcation leading to unstable limit cycles for a linearly stable system. However, for the conditions examined so far, it did not produce the right type of coupling to nonlinear gasdynamics to produce a fold that could lead to stable limit cycles. Hence, this term does not lead to triggering. Furthermore, for reasonable values of R_b, term C had little effect on the system and can be neglected if we restrict ourselves to perturbations of 15% or less.

Of the three terms in the Baum and Levine model, term B is the only first-order term. As a result, this term is dominant in many cases of interest. Figure 5 compares results for term A only, term B only, and all three terms for truncation to two modes. At first it appears that term B does not produce good results for this case. However, by comparison with the four-mode case shown in Fig. 6, it is clear that results for term B are actually closer to the four-mode case. There is even better agreement for the four-mode case with only term B. Therefore, to study the effects of truncation and time-averaging, only this term will be used for the nonlinear combustion model. Note that of the three contributions defined here, the term B is closest to representing the most familiar hypothesis for the mechanism of velocity coupling, namely kinematical rectification.

B. Effects of Truncation

When solving Eqs. (10), it is necessary to truncate the system of equations to a finite number of modes. For the purposes of the approximate analysis, it is desirable to use the smallest number of modes that still provides accurate results. Many previous studies involving nonlinear gasdynamics only have used truncation to two modes, the minimum number of modes required to produce a limit cycle. As Jahnke and Culick[23] have shown, this approximation does not always produce good results for highly unstable systems. This approximation may become invalid for linearly stable systems as well when nonlinear combustion is also taken into account.

Figure 7 shows results for two-, four-, and six-mode approximations with $\bar{R}_{vc}$ = 5.32. For this value of $\bar{R}_{vc}$, all three cases give the same qualitative behavior, although quantitatively, the two-mode case is not satisfactory and produces much larger amplitudes. The four-mode case, however, agrees well with the six-mode case, as long as the linear growth rate of the fundamental mode is not too large.

In general, the system of equations appears to be more sensitive to truncation errors when nonlinear combustion is included. Depending on the strength and the form of the nonlinear combustion model, more modes may be required to obtain accurate results. It does not appear that the required number of modes can be predicted in advance. Instead, this depends on the parameters of the combustion response, as well as all parameters of the system. Although these results suggest that in many cases, truncation to four modes can yield quite satisfactory results, care must be taken to pay attention always to the possible consequences of truncation, as Jahnke and Culick have shown.

C. Effects of Time-Averaging

The method of time-averaging as described in Sec. II has been used often in previous works. In recent studies con-

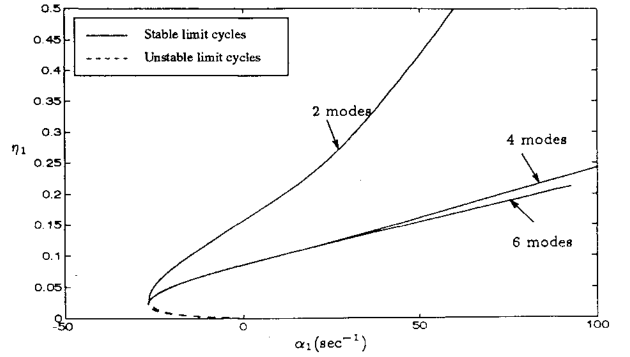

Fig. 7 Maximum amplitude of first acoustic mode in limit cycle using term B with the original oscillator equations; two, four, and six modes; $\bar{R}_{vc}$ = 5.32.

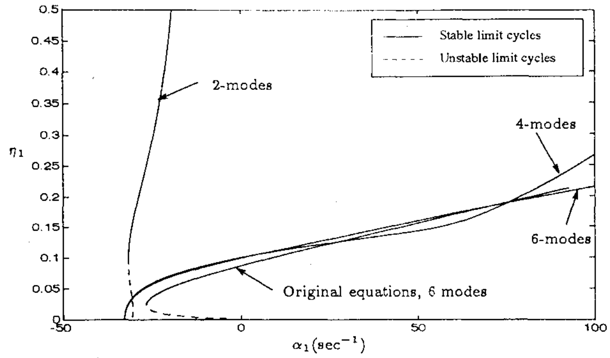

Fig. 8 Maximum amplitude of first acoustic mode using term B with the time-averaged equations; two, four, and six modes; $\bar{R}_{vc}$ = 5.32.

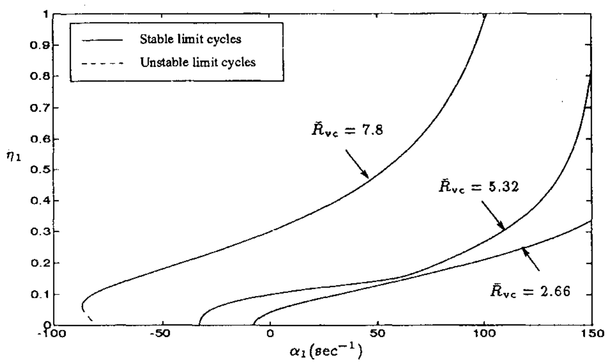

Fig. 9 Maximum amplitude of first acoustic mode in limit cycle using term B with the time-averaged equations and various values of $\bar{R}_{vc}$; four modes.

cerned with nonlinear gasdynamics only, the effects of time-averaging were investigated by Jahnke and Culick[23] for two modes, and by Burnley[26] for higher numbers of modes. It was determined that for the two-mode case, results obtained using time-averaging were only valid for mildly unstable systems. By including more modes, the method of time-averaging can also produce good agreement with the original oscillator equations for more unstable systems. In the previous studies of triggering by Kim[18] and Greene,[19] two-mode, time-averaged equations were used to show a limited amount of triggering. It has already been shown that truncation can have a strong effect when nonlinear combustion is present, and we will now show that time-averaging can have a substantial effect as well.

A useful simplification when applying the method of time-averaging to the nonlinear combustion model is to assume that $\text{sign}(u') = -\text{sign}(\dot{\eta}_1)$, based on the observation that the amplitude of the fundamental mode is usually greater than the amplitude of higher modes. Using this simplification, the method of time-averaging can be applied to the nonlinear combustion model in much the same manner as the nonlinear gasdynamics terms. The main difference is that the integral in Eqs. (13a) and (13b) must be split into two parts due to the discontinuity of the sign function. Greene[19] has a more thorough description of applying time-averaging to this model.

Using time-averaged equations, the possibility of triggering has also been shown. The range of α_1 for which triggering occurs is, however, much smaller than the original oscillator equations and depends on the number of modes included, as well as the value of $\bar{R}_{vc}$. Figure 8 shows results obtained using the time-averaged equations for two, four, and six modes and a value of $\bar{R}_{vc}$ = 5.32. For comparison, the six-mode case of the original oscillator equations is also included. Of these three time-averaged cases, only truncation to two modes produces the possibility of triggering. However, when compared with results for the original oscillator equations, it is neither qualitatively nor quantitatively accurate. The four- and six-mode cases produce a supercritical bifurcation at $\alpha_1 \approx 34\ \text{s}^{-1}$, which is essentially a shift in α due to the linear self-coupling of term B. None of the results are satisfactory when compared to the six-mode case for the original oscillator equations. For higher values of $\bar{R}_{vc}$, the four- and six-mode cases also produce the possibility of triggering, although still for a smaller range of α_1 than the original equations. An example of the four-mode case is shown in Fig. 9 for $\bar{R}_{vc}$ = 2.66, 5.32, and 7.8. A small region of triggering is found for $\bar{R}_{vc}$ = 7.8, although for a much smaller range of α_1 than the region predicted by the original oscillator equations.

VI. Concluding Remarks

The approximate analysis summarized in Sec. II provides a framework to study nonlinear acoustics in combustion chambers. When used in conjunction with dynamical systems theory, this approximate analysis provides a basis for determining general trends, in particular the possibility of pulsed instabilities. Previous works have shown convincingly that nonlinear gasdynamics alone does not produce the possibility of triggering to stable limit cycles. Another nonlinear process is required. Baum and Levine first used numerical solution to the one-dimensional equations to show that pulsed instabilities could be found when nonlinear combustion is accounted for. Later, Greene demonstrated the same result with the time-averaged equations of the approximate analysis.

Using the two ad hoc models chosen by Baum and Levine, and by Greene, the present work has shown the existence of pulsed instabilities when nonlinear combustion is included in the approximate analysis, with or without time-averaging. The two models produced similar results, since they differ only by term A of Eq. (33). Although it is only second-order, this term causes Baum and Levine's model to be more sensitive to truncation: more modes must be included in order to obtain good accuracy.

Two useful approximations used extensively in the past are the method of time-averaging and truncation to a small number of modes. The effects of these approximations with nonlinear combustion were also investigated. When only nonlinear gasdynamics are included, the two-mode, time-averaged equations can produce accurate results for some mildly unstable cases. This does not appear to be the case, however, when nonlinear combustion is also included. Both truncation to two modes and time-averaging can seriously affect the accuracy of results; especially, time-averaging should be used with care when nonlinear combustion is included.

Acknowledgments

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